ME 305 Fluid Mechanics I

Part 6

Differential Formulation of Fluid Flow

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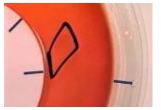
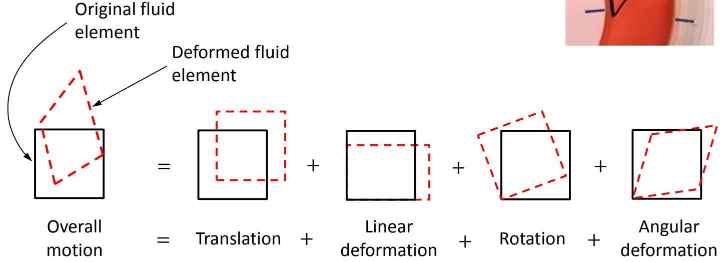
Motion of a Fluid Element (Fluid Kinematics)

- In a general flow field, fluid motion can be decomposed into the following 4 components

1) translation
2) linear deformation
3) rotation
4) angular deformation

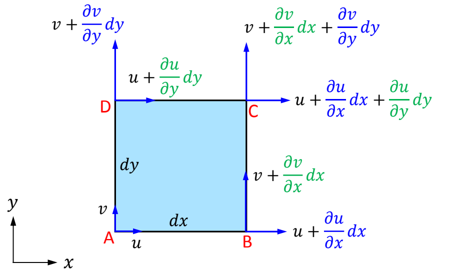
Movie :
Fluid deformation





Fluid Kinematics (cont'd)

- Consider the following 2D, differential fluid element with corner A moving with a velocity of $u\vec{i} + v\vec{j}$.
- Velocity components of the other corners can be determined as follows using first-order Taylor series approximation.

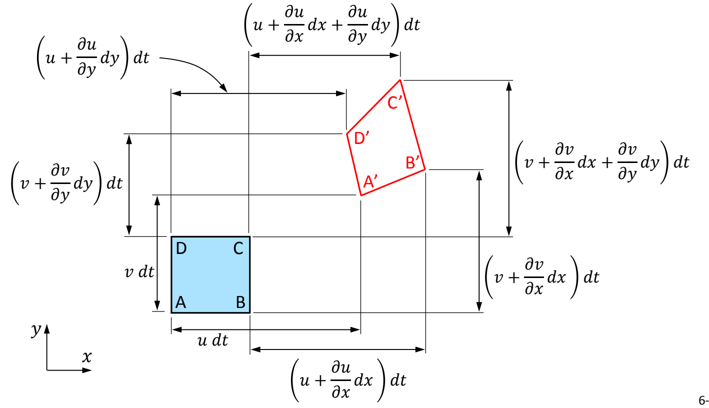


Color Code

Translation
Linear deformation
Rotation & angular deformation

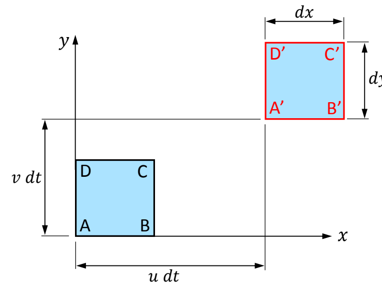
Fluid Kinematics (cont'd)

- Due to different velocities of each corner, fluid element will move and deform in a small dt time.



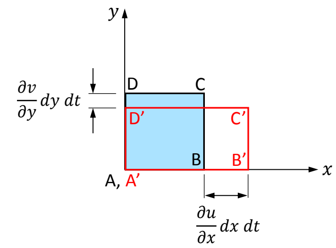
1) Translation

- Only the position of the fluid element changes. Its size, orientation and shape remain the same.
- All corners are moving with the same u and v velocity.
- Below, both u and v are shown to be positive, but it is not necessarily the case.



2) Linear Deformation

- Only the size of the fluid element changes. Its position, orientation and shape remain the same.
- Corner A is fixed, because all its motion was previously considered in translation.
- Corner B moves in x direction only and corner D moves in y direction only.
- Below, $\partial u/\partial x$ and $\partial v/\partial y$ are shown to be positive and negative, respectively, but it is not necessarily the case.



2) Linear Deformation (cont'd)

- When we extend linear deformation to 3D, size changes in x, y and z directions are

$$\frac{\partial u}{\partial x} dx\, dt \quad , \quad \frac{\partial v}{\partial y} dy\, dt \quad , \quad \frac{\partial w}{\partial z} dz\, dt$$

- These values can be positive or negative.

- Size changes can also be expressed as linear strains as ''Size change / Original size''

$$\varepsilon_x = \frac{\frac{\partial u}{\partial x} dx\, dt}{dx} = \frac{\partial u}{\partial x} dt \quad , \quad \varepsilon_y = \frac{\frac{\partial v}{\partial y} dy\, dt}{dy} = \frac{\partial v}{\partial y} dt \quad , \quad \varepsilon_z = \frac{\frac{\partial w}{\partial z} dz\, dt}{dz} = \frac{\partial w}{\partial z}\, dt$$

- Comparison of initial and final volume of a 3D fluid element provides an important quantity called dilation.

2) Linear Deformation (cont'd)

- Initial volume of fluid element in 3D : $\forall_i = dx\, dy\, dz$
- Final volume of fluid element :

$$\forall_f = \left(dx + \frac{\partial u}{\partial x} dx\, dt\right)\left(dy + \frac{\partial v}{\partial y} dy\, dt\right)\left(dz + \frac{\partial w}{\partial z} dz\, dt\right)$$

$$\forall_f \approx \left[1 + \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) dt\right] dx\, dy\, dz$$

Neglected terms are very small compared to the kept ones

- Dilation is the rate of change of volume per initial volume

Divergence of the velocity field

$$\text{Dilation} = \frac{\frac{\forall_f - \forall_i}{dt}}{\forall_i} = \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = \nabla \cdot \vec{V}$$

2) Linear Deformation (cont'd)

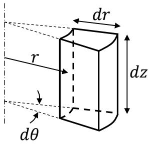
- Dilation ($\nabla \cdot \vec{V}$) is related to the compressibility of the flow.
- For incompressible flows dilation is zero, i.e. fluid element's size can not change.

Incompressible $\rightarrow$ $\nabla \cdot \vec{V} = 0$

Exercise : In the cylindrical coordinate system divergence of velocity is

$$\nabla \cdot \vec{V} = \frac{1}{r}\frac{\partial (rV_r)}{\partial r} + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{\partial V_z}{\partial z}$$

Repeat the dilation calculation of the previous slide for a differential fluid element in cylindrical coordinate system and see if you can get the above result or not.

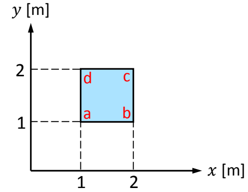


6-9

2) Linear Deformation (cont'd)

Exercise : The velocity field $\vec{V} = 0.3x\vec{i} - 0.3y\vec{j}$ represents the flow turning at a 90º corner. A square is marked in the fluid as shown in $t = 0$. Evaluate the new position of four corner points when corner "a" has moved to $x = 1.5$ m after τ seconds.

- Evaluate the size changes and linear strains in x and y directions.
- Calculate area change and dilation of the element.
- Is this an incompressible flow?



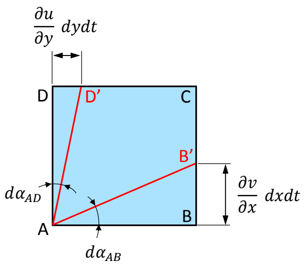
6-10

3) & 4) Combined Rotation and Angular Deformation

- Due to combined rotation and angular deformation, sides AB and AD will rotate as shown.

$$\tan(d\alpha_{AB}) \approx d\alpha_{AB} = \frac{\frac{\partial v}{\partial x}dxdt}{dx} = \frac{\partial v}{\partial x}dt$$

$$\tan(d\alpha_{AD}) \approx d\alpha_{AD} = \frac{\frac{\partial u}{\partial y}dydt}{dy} = \frac{\partial u}{\partial y}dt$$



- Angular speeds of sides AB and AD are

$$\omega_{AB} = \frac{d\alpha_{AB}}{dt} = \frac{\partial v}{\partial x}$$

$$\omega_{AD} = -\frac{d\alpha_{AD}}{dt} = -\frac{\partial u}{\partial y}$$

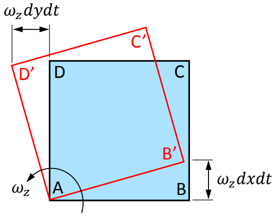
Line AD rotates CW if $\partial u/\partial y$ is positive. But a CW angular speed should be negative. Minus sign is added for this purpose.

6-11

3) Rotation

- Rate of rotation of fluid element ABCD about the z-axis is defined as the average of the angular speeds of two mutually perpendicular lines AB and AD.

$$\omega_z = \frac{1}{2}(\omega_{AB} + \omega_{AD}) = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$



- For a 3D flow field angular speeds around x and y axes are defined in a similar way

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) , \qquad \omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

6-12

3) Rotation (cont'd)

- Angular velocity vector is defined as

$$\vec{\omega} = \frac{1}{2}\left[\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\vec{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\vec{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k}\right] = \frac{1}{2}\underbrace{\nabla \times \vec{V}}_{\text{Curl of velocity}}$$

- In cylindrical coordinate system

$$\vec{\omega} = \frac{1}{2}\nabla \times \vec{V} = \frac{1}{2}\left[\left(\frac{1}{r}\frac{\partial V_z}{\partial \theta} - \frac{\partial V_\theta}{\partial z}\right)\vec{i_r} + \left(\frac{\partial V_r}{\partial z} - \frac{\partial V_z}{\partial r}\right)\vec{i_\theta} + \frac{1}{r}\left(\frac{\partial (rV_\theta)}{\partial r} - \frac{\partial V_r}{\partial \theta}\right)\vec{i_z}\right]$$

- Vorticity of a flow field is defined as

$$\vec{\xi} = 2\vec{\omega} = \nabla \times \vec{V}$$

- For an irrotational flow vorticity (or angular velocity, or curl of velocity) is zero everywhere in the flow field.

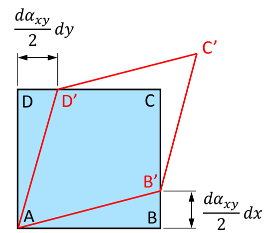
6-13

4) Angular Deformation

- Angular deformation is related to the rate of change of the right angle between sides AB and AD, which is

$$\frac{d\alpha_{xy}}{dt} = \frac{d\alpha_{AB}}{dt} + \frac{d\alpha_{AD}}{dt} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

where α_{xy} is the shear strain in the xy plane.



- For a 3D flow field rate of shear strains in yz and xz planes can be defined in a similar way

$$\frac{d\alpha_{yz}}{dt} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}$$

$$\frac{d\alpha_{xz}}{dt} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

6-14

4) Angular Deformation (cont'd)

- Remember that for a Newtonian fluid shear stress is proportional to the rate of shear strain. For a flow in the xy plane

$$\tau_{xy} = \mu\frac{d\alpha_{xy}}{dt} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

This 2nd term was zero for the "flow between parallel plates" example that was studied in Part 1. In a general flow field it is not necessarily zero.

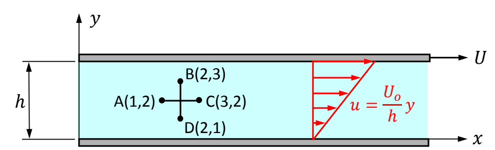
- There is a direct link between shear stress and rotationality.
  - Pressure or body forces can not rotate a fluid element.
  - But shear forces can create rotation.
  - Shear (viscous) forces are especially important close to solid boundaries.
  - Away from the solid boundaries flow may be assumed irrotational.
  - A totally irrotational flow is an idealization, which can not exist in real life. But it is still a very useful assumption.

6-15

Exercises for Kinematics of Fluid Flow

Exercise : Combine the translation, linear deformation, rotation and angular deformation that we studied separately in the previous slides and show that the new positions of corners B and D of the square fluid element ABCD are actually the ones shown in slide 6-4.

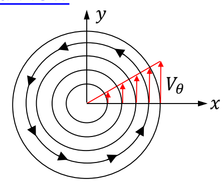
Exercise : Velocity field for the flow in a narrow gap is given as $\vec{V} = \frac{Uy}{h}\vec{i}$ where $U = 4$ mm/s and $h = 4$ mm. At $t = 0$ the segments AC and BD are marked to form a cross. Determine the positions of the marked points at $t = 1.5$ s. Calculate the rate of angular deformation and rate of rotation of a fluid particle in this flow field (Reference: Fox's book).



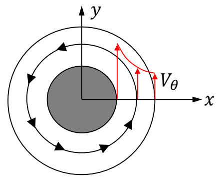
6-16

Exercises for Kinematics of Fluid Flow

Exercise : Fluid particles in a 2D flow field are rotating in circular paths about the z-axis at a constant angular velocity of ω, as if they were a rigid body. The velocity field is given as $\vec{V} = \omega r\, \vec{i_\theta}$. This flow is known as a forced vortex. Is this a rotational flow field ?



Exercise : Fluid particles in a 2D flow field are rotating in circular paths according to the following velocity field $\vec{V} = \frac{c}{r}\, \vec{i_\theta}$. This flow is known as a free vortex. Is this a rotational flow field ?



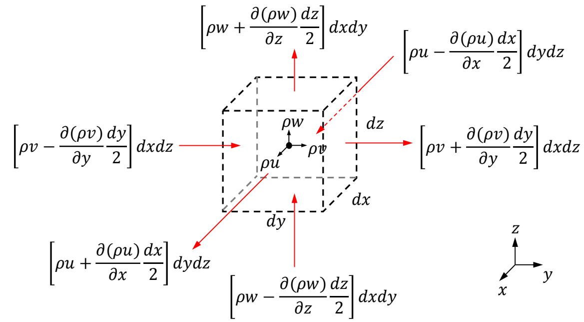
Exercise : In Slide 6-15 τ_{xy} is given for a flow in the xy plane. Use your fluid mechanics book to find the expression for $\tau_{r\theta}$ for a flow in the $r\theta$ plane. Calculate $\tau_{r\theta}$ for the flow fields given in the above exercises. Are there angular deformation in these flows?

Differential Formulation of Continuity Equation

- Consider the following infinitesimal control volume. At its centroid

$$\rho\vec{V} = \rho u\vec{i} + \rho v\vec{j} + \rho w\vec{k}$$

- Mass flow rates passing through the faces can be determined using first order TSE.



Continuity Equation (cont'd)

- Add all the mass fluxes through the faces to get net mass outflow per unit time

$$\left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right]dxdydz$$

- This net mass outflow rate should be balanced with the following rate of change of mass in the differential CV

$$\frac{\partial m}{\partial t} = \frac{\partial \rho}{\partial t}dxdydz$$

resulting in the continuity equation in differential form

$$\underbrace{\frac{\partial \rho}{\partial t}dxdydz}_{\text{Rate of change of mass within the differential CV}} + \underbrace{\left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right]dxdydz}_{\text{Net mass outflow per unit time}} = 0$$

Continuity Equation (cont'd)

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \quad \rightarrow \quad \boxed{\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{V}) = 0}$$

- Opening up the dot product

$$\underbrace{\frac{\partial \rho}{\partial t} + (\vec{V}\cdot\nabla\rho)}_{\frac{d\rho}{dt}} + \rho(\nabla\cdot\vec{V}) = 0$$

(using $\frac{dN}{dt} = \frac{\partial N}{\partial t} + (\vec{V}\cdot\nabla)N$ for density)

$$\boxed{\frac{d\rho}{dt} + \rho(\nabla\cdot\vec{V}) = 0} \quad \rightarrow \quad \frac{d\rho}{dt} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0$$

- For a steady flow, using the first boxed equation : $\nabla\cdot(\rho\vec{V}) = 0$
- For an incompressible flow, using the second boxed equation : $\nabla\cdot\vec{V} = 0$

Exercises for Continuity Equation

Exercise : Velocity components for a certain incompressible steady flow are as follows. Determine w (Munson's book).

$$u = x^2 + y^2 + z^2$$

$$v = xy + yz + z$$

$$w = ?$$

Exercise : For an incompressible flow, even if the flow is unsteady no time derivative remains in the continuity equation. What does this mean physically?

Exercise : Consider a differential CV in the cylindrical coordinate system, similar to the one given in Slide 6-9. Using the procedure described in Slides 6-18 and 6-19 derive the following continuity equation in the cylindrical coordinate system.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = \frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial (r \rho V_r)}{\partial r} + \frac{1}{r}\frac{\partial (\rho V_\theta)}{\partial \theta} + \frac{\partial (\rho V_z)}{\partial z} = 0$$

Streamfunction (ψ (psi))

- Streamfunction is a mathematical tool that can be used to define a flow field using a single scalar instead of multiple velocity components.
- It can be defined for "2D incompressible" or "2D steady" flows.
- Consider the 2D incompressible flow in the xy plane case.
- Continuity equation : $\nabla \cdot \vec{V} = 0 \quad \rightarrow \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ (for a flow in the xy plane)
- Velocity field is defined by two components u and v, but they are related via the continuity equation.
- If we define a function $\psi\,(x, y)$ as

$$u = \frac{\partial \psi}{\partial y} \quad , \quad v = -\frac{\partial \psi}{\partial x}$$

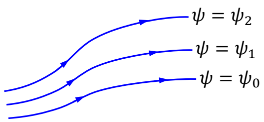
continuity equation is automatically satisfied, i.e.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right) = \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial x \partial y} = 0$$

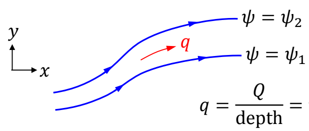
Streamfunction (cont'd)

- Therefore for a "2D incompressible" flow instead of working with two velocity components, it is possible to work with a single variable called the streamfunction.

Exercise : Show that streamfunction has a constant value on a streamline.



Exercise : Show that for a 2D incompressible flow, volumetric flow rate per unit depth between any two streamlines is equal to difference between the streamfunctions defining these streamlines.



$$q = \frac{Q}{\text{depth}} = \psi_2 - \psi_1$$

Streamfunction (cont'd)

Exercise : The velocity components in a flow field are $u = 0$, $v = -y^3 - 4z$ and $w = 3y^2 z$. Is this a 2D flow? Is this an incompressible flow? If possible determine the streamfunction. Is the flow rotational?

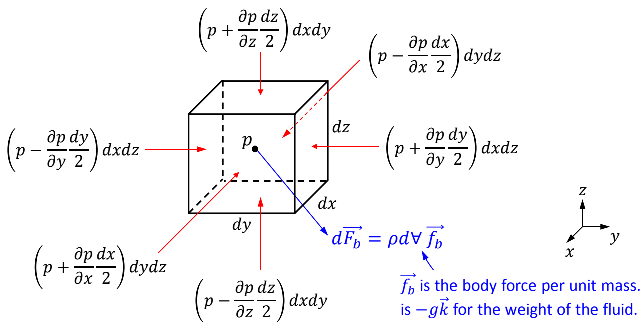
Exercise : Consider a steady, but not necessarily incompressible flow in the xy plane. What should be the relations (similar to the ones in the box of Slide 6.22) between the streamfunction and the velocity components so that the continuity equation is exactly satisfied. Hint: Continuity equation to be used is $\nabla \cdot (\rho \vec{V}) = 0$ and now density is also a part of the formulation.

Exercise : Consider an incompressible flow in the $r\theta$ plane of the cylindrical coordinate system. What should be the relations between the streamfunction and the velocity components so that the continuity equation is exactly satisfied. Hint: Use the continuity equation of slide 6-21.

Repeat this exercise if the flow is in the rz plane.

Euler's Equation of Motion

- Euler's equation is the differential form of linear momentum conservation for inviscid flows.
- To derive it consider the pressure and body forces acting on a differential fluid element. Pressure is taken as p at the element's centroid.



$\vec{f_b}$ is the body force per unit mass. It is $-g\vec{k}$ for the weight of the fluid.

6-25

Euler's Equation (cont'd)

- Sum of all the forces will accelerate the fluid element as follows

$$\sum d\vec{F} = dm\,\vec{a} \quad \rightarrow \quad (-\nabla p)\, d\forall + \rho\, d\forall\, \vec{f_b} = \rho\, d\forall\, \vec{a}$$

$$\boxed{\vec{a} = \vec{f_b} - \frac{1}{\rho}\nabla p} \quad \text{or} \quad \boxed{\frac{\partial \vec{V}}{\partial t} + (\vec{V}\cdot\nabla)\vec{V} = \vec{f_b} - \frac{1}{\rho}\nabla p}$$

- Three components of the Euler's equation in the Cartesian coordinate system are

$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = f_{b_x} - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$a_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = f_{b_y} - \frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$a_z = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = f_{b_z} - \frac{1}{\rho}\frac{\partial p}{\partial z}$$

Typically $f_{bx} = f_{by} = 0$ and $f_{bz} = -g$

6-26

Euler's Equation (cont'd)

- Three components of the Euler's equation in the cylindrical coordinate system are

$$a_r = \frac{\partial V_r}{\partial t} + V_r\frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta} - \frac{V_\theta^2}{r} + V_z\frac{\partial V_r}{\partial z} = f_{b_r} - \frac{1}{\rho}\frac{\partial p}{\partial r}$$

$$a_\theta = \frac{\partial V_\theta}{\partial t} + V_r\frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_\theta}{\partial \theta} - \frac{V_r V_\theta}{r} + V_z\frac{\partial V_\theta}{\partial z} = f_{b_\theta} - \frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial \theta}$$

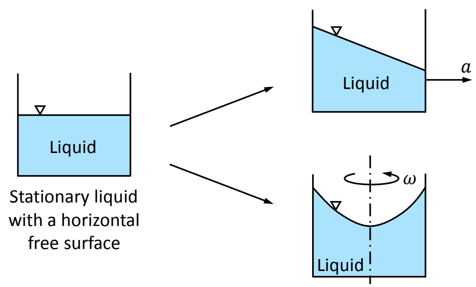
$$a_z = \frac{\partial V_z}{\partial t} + V_r\frac{\partial V_z}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_z}{\partial \theta} + V_z\frac{\partial V_z}{\partial z} = f_{b_z} - \frac{1}{\rho}\frac{\partial p}{\partial z}$$

Exercise : x component of the velocity in a 2D, incompressible, irrotational, frictionless flow is given as $u = 6x$. At point (2,0,0) the y component of velocity is known to be zero. $w = 0$ everywhere. Body force is $-g\vec{k}$. Obtain an expression for v. Find the acceleration at point (2,0,0). Obtain an expression for the pressure field if the pressure is known to be p_0 at (0,0,0).

6-27

Use of Euler's Equation for Fluids in Rigid Body Motion

- Consider a body of fluid that is in rigid body motion, i.e. it moves as if it is a solid body with fluid particles having no relative motion with respect to each other.
- In such a case, fluid is free of shear stress.
- Two examples of this are fluids moving with constant linear acceleration and fluids rotating around an axis with constant angular velocity.



Constant linear acceleration causes a tilted planar free surface.

Rotation at constant speed causes a curved (parabolic) free surface.

6-28

Fluids in Rigid Body Motion (cont'd)

- For a fluid in rigid body motion, there are no viscous forces.
- There are only pressure and body forces, similar to a static fluid.
- But the difference is, now we have nonzero acceleration.
- Euler's equation is valid for fluids in rigid body motion.

$$\vec{a} = \vec{f_b} - \frac{1}{\rho}\nabla p$$

$\vec{a}$ was zero for a static fluid. But here it is not zero.

Typically $\vec{f_b} = \vec{g}$

- Nonzero acceleration will cause a change in the pressure distribution, compared to a static fluid.

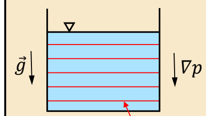
Fluids in Rigid Body Motion (cont'd)

$$\vec{a} = \vec{g} - \frac{1}{\rho}\nabla p$$

For a static fluid

$\vec{a} = 0$

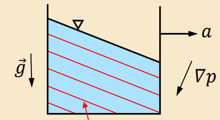
$\nabla p = \rho\vec{g}$



For a fluid moving with constant linear acceleration

$\vec{a} =$ constant

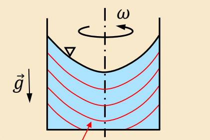
$\nabla p = \rho(\vec{g} - \vec{a})$



For a fluid rotating at constant speed

$\vec{a} = a_r\,\vec{i}_r$

$\nabla p = \rho(\vec{g} - \vec{a})$



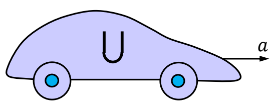
Red lines are constant pressure lines. ∇p is perpendicular to them.

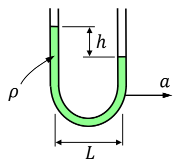
Fluids in Rigid Body Motion (cont'd)

Exercise : We want to use the following U-tube filled with a liquid as a crude accelerometer in our car. As the car speeds up with constant acceleration we should observe a difference between the levels of the two liquid columns. Determine

a) the pressure distribution inside the liquid

b) relation between acceleration and the parameters of our device.





Fluids in Rigid Body Motion (cont'd)

Exercise : A cylindrical tank, with its top open to atmospheric pressure has a Radius of $R = 0.5$ m and a height of $H = 2$ m. It is completely filled with water and rotated about its axis at an angular velocity of $\omega = 5$ rad/s. Determine

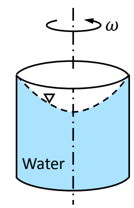
a) the pressure distribution inside the water

b) the pressures at $(r = 0, z = 0)$ and $(r = 0.5\text{ m}, z = 0)$

c) the pressure distribution on the side wall of the tank

d) the force exerted by the water on the bottom of the tank

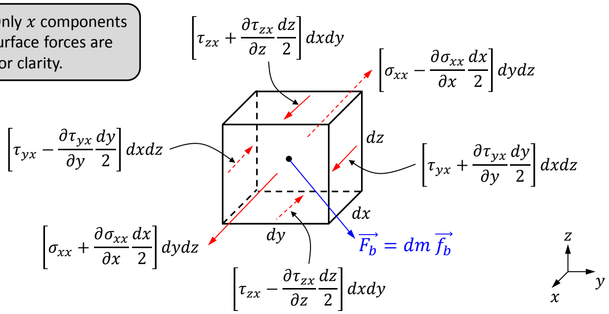
e) the volume of the spilled water.



Navier-Stokes Equation

- Navier-Stokes equation is the differential form of linear momentum conservation for viscous flows. It is Newton's second law written for fluid flow.
- Pressure, viscous and body forces need to be considered.

Note : Only x components of the surface forces are shown for clarity.



Navier-Stokes Equation (cont'd)

- Add all the forces and substitute the sum into Newton's 2nd Law of Motion.
- Express normal and viscous stresses in terms of pressure and velocity components.
- Skipping the details (you are NOT responsible for them), for a Newtonian fluid with constant fluid properties (viscosity and density), we get

$$\vec{a} = \boxed{\underbrace{\frac{\partial \vec{V}}{\partial t} + (\vec{V}\cdot\nabla)\vec{V} = \vec{f_b} - \frac{1}{\rho}\nabla p}_{\text{Euler's Equation}} + \nu\nabla^2\vec{V}}$$

Additional viscous term.

$\nu = \mu/\rho$: Kinematic viscosity

∇^2 : Laplace operator

- Navier-Stokes equation can also be written in terms of dynamic viscosity by multiplying all the terms with density

$$\rho\left(\frac{\partial \vec{V}}{\partial t} + (\vec{V}\cdot\nabla)\vec{V}\right) = \rho\vec{f_b} - \nabla p + \mu\nabla^2\vec{V}$$

Navier-Stokes Equation (cont'd)

- Components in the Cartesian coordinate system are

$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = f_{b_x} - \frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\overbrace{\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)}^{\nabla^2 u}$$

$$a_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = f_{b_y} - \frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)$$

$$a_z = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = f_{b_z} - \frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

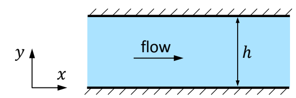
- These equations can be solved analytically only for a few simple geometries and boundary conditions, such as the ones given in the exercises of the coming slides.

Exercise : Find the three components of the Navier-Stokes equations in cylindrical coordinate system from a fluid mechanics textbook and write them at the back of this slide for future reference.

Poiseuille Flow

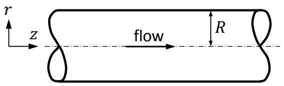
Exercise : Consider the flow between two infinitely wide parallel plates driven by a pressure gradient in the axial direction. Neglect body forces and consider steady, laminar, incompressible flow of a Newtonian fluid with constant viscosity and density. Simplify the continuity and Navier-Stokes equations and determine

- velocity profile
- shear stress distribution
- volumetric flow rate
- maximum and average velocities
- pressure drop over a length of L.



Hagen-Poiseuille Flow

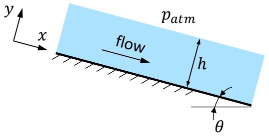
Exercise : Repeat the exercise of the previous slide for the steady, incompressible flow inside a constant diameter pipe.



Flow Down an Inclined Plane

Exercise : An incompressible fluid flows down an inclined plane in a steady, fully developed laminar film of thickness h. This time fluid weight is not negligible. Simplify the continuity and N-S equations for this flow and study the flow field in detail.

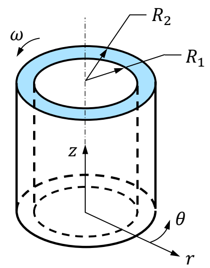
Hint: At the free surface consider that the air applies negligible shear force to the liquid.



Flow Between Concentric Cylinders

Exercise : A liquid fills the annular gap between vertical concentric cylinders. Inner cylinder is stationary and outer one rotates at constant speed. Simplify the continuity and Navier-Stokes equations for this steady, laminar flow and determine

- the velocity profile
- shear stress distribution



Compare the shear stress at the surface of the inner cylinder with that computed from a planar approximation obtained by "unwrapping" the annulus into a plane and assuming a linear velocity profile across the gap. Determine the ratio of cylinder radii for which the planar approximation predicts the correct shear stress at the surface of the inner cylinder within 1 % accuracy.

(Reference: Fox's book)